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Efficient option pricing with path integral

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Abstract. An efficient computational algorithm to price financial derivatives is presented. It is based on a path integral formulation of the pricing problem. It is shown how the path integral approach can be worked out in order to obtain fast and accurate predictions for the value of a large class of options, including those with path-dependent and early exercise features. As examples, the application of the method to European and American options in the Black-Scholes model is illustrated. The results of the algorithm are compared with those obtained with the standard procedures known in the literature and found to be in good agreement.

PACS. 02.50.Ey Stochastic processes – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 89.75.-k Complex systems

1 Introduction

The standard theory of option pricing is based on the results found in 1973 by Black and Scholes [1] and, independently, Merton [2]. Their pioneering work starts from the basic assumption that the asset prices follow the dynamics of a particular stochastic process, so that they have a log-normal distribution [3,4]. In the case of an efficient market with no arbitrage possibilities, no dividends and constant volatilities, they found that the price of each financial derivative is ruled by an ordinary partial differential equation, known as the Black-Scholes-Merton (BSM) formula. In the most simple case of a so-called European option, the BSM equation can be explicitly solved to obtain an analytical formula for the price of the option. When we consider other financial derivatives, which are commonly traded in real markets and allow anticipated exercise and/or depend on the history of the underlying asset, the BSM formula fails to give an analytical result. Appropriate numerical procedures have been developed in the literature to price exotic financial derivatives with path-dependent features, as discussed in references [3,5]. The aim of this work is to provide a contribution to the problem of efficient option pricing in financial analysis, showing how it is possible to use path integral methods to develop a fast and precise algorithm for the evaluation of option prices.

The path integral method, which traces back to the original work of Wiener and Kac in stochastic calculus [6,7] and of Feynman in quantum mechanics [8], is today widely employed in chemistry and physics, and very recently in finance too [9–13], because it gives the possibility of applying powerful analytical and numerical techniques [14]. Following recent studies on the application of the path integral approach to the financial market as appeared in the econophysics literature (see Ref. [13] for a comprehensive list of references), this paper is devoted to present an original, efficient path integral algorithm to price financial derivatives, including those with path-dependent and early exercise features, and to compare the results with those obtained with the standard procedures known in the literature. The paper is a short version of reference [15], to which the reader is referred for more details and further numerical results.

The outline of the paper is as follows. In Section 2 the path integral approach to option pricing is briefly reviewed and analytically developed in order to obtain an efficient procedure for the calculation of the transition probability associated to a given stochastic model of asset evolution. Computational details to obtain fast predictions for path-dependent options are also described, and a particularly simple and very quick semi-analytical approximation for the price of an American option is derived, by exploiting the possibility of anticipated exercise for any time before the expiration date. A sample of numerical results for European and American options in the BSM model is given in Section 3, together with comparisons with results known in the literature. Conclusions and prospects are drawn in Section 4.

2 The path integral method

The path integral method is an integral formulation of the dynamics of a stochastic process. It is a suitable framework for the calculation of the transition probabilities associated to a given stochastic process, which is seen as the

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convolution of an infinite sequence of infinitesimal short-time steps [9,14].

For definiteness, the stochastic model here assumed for the time evolution of the asset price S is the standard BSM geometric Brownian motion, driven by the stochastic differential equation [3,4]

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}w, \tag{1}$$

which, by means of the Itô lemma, can be cast in the form of an arithmetic Brownian motion for the logarithm of S

$$\mathrm{d}(\ln S) = A\mathrm{d}t + \sigma \mathrm{d}w, \tag{2}$$

where σ is the volatility, $A \doteq (\mu - \sigma^2/2)$, μ is the drift parameter and w is the realization of a Wiener process such that, for a time interval $\mathrm{d}t$, it satisfies the statistical properties $\langle \mathrm{d}w \rangle = 0$ and $\langle \mathrm{d}w^2 \rangle = \mathrm{d}t$. For the problem of option pricing, the path integral method can be successfully employed for the explicit calculation of the expectation values of the quantities of financial interest, given by integrals of the form

$$E[\mathcal{O}_i | S_{i-1}] = \int \mathrm{d}z_i p(z_i|z_{i-1}) \mathcal{O}_i(\mathrm{e}^{z_i}). \tag{3}$$

In equation (3) $z = \ln S$ and $p(z_i|z_{i-1}) \doteq p(z_i, t_i| z_{i-1}, t_{i-1})$ [1] denotes the conditional transition probability to have at the time t_i a price $z_i = \ln S_i$ under the hypothesis that the price was $z_{i-1} = \ln S_{i-1}$ at a previous time $t_{i-1} < t_i$. $E[\mathcal{O}_i|S_{i-1}]$ is the conditional expectation value of some functional $\mathcal{O}_i$ of the stochastic process. For example, for an European call option at the maturity T the quantity of interest will be $\max\{S_T - X, 0\}$, X being the strike price.

As discussed in the literature [3,5,10,13], the computational complexity associated to this calculation is generally great: in the case of exotic options, with path-dependent and early exercise features, integrals of the type (3) can not be analytically solved. As a consequence, we demand two things from a path integral framework: a very quick way to estimate the transition probability associated to a given stochastic model and a clever choice of the integration points with which evaluate the integrals (3).

2.1 Transition probability

The probability distribution function related to a stochastic differential equation verifies the so-called Chapman-Kolmogorov equation [4]

$$p(z''|z') = \int \mathrm{d}z p(z''|z) p(z|z'), \tag{4}$$

which states that the probability (density) of a transition from the value z' (at time t') to the value z'' (at time t'') is the "summation" over all the possible intermediate values z of the probability of separate and consequent transitions $z' \to z$, $z \to z''$. As a consequence, if we consider a finite-time interval $[t', t'']$ and we apply a time slicing, by considering $n+1$ subintervals of length $\Delta t \doteq (t'' - t')/n + 1$, we can write, by iteration of equation (4)

$$p(z''|z') = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}z_1 \cdots \mathrm{d}z_n p(z''|z_n) \cdots p(z_1|z'),$$

which, thanks to the formula valid for the transition probability $z_i \to z_f$ associated to the process (2) for a time interval Δt [4,9],

$$p(z_f|z_i) = \frac{1}{\sqrt{2\pi \Delta t \sigma^2}} \exp\left\{ -\frac{[z_f - (z_i + A\Delta t)]^2}{2\sigma^2, \Delta t} \right\} \tag{5}$$

can be rewritten as

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}z_1 \cdots \mathrm{d}z_n \frac{1}{\sqrt{(2\pi\sigma^2 \Delta t)^{n+1}}} \times \exp\left\{ -\frac{1}{2\sigma^2 \Delta t} \sum_{k=1}^{n+1} [z_k - (z_{k-1} + A\Delta t)]^2 \right\}. \tag{6}$$

In the limit $n \to \infty$, $\Delta t \to 0$ such that $(n+1)\Delta t = (t'' - t')$, the previous expression, as explicitly shown in reference [9], exhibits a Lagrangian structure and it is possible to express the transition probability in the path integral formalism as a convolution of the form [9]

$$p(z''|z') = \int_{\mathcal{C}} \mathcal{D}[\sigma^{-1}\tilde{z}] \times \exp\left\{ -\int_{t'}^{t''} L(\tilde{z}(\tau), \dot{\tilde{z}}(\tau); \tau) \mathrm{d}\tau \right\},$$

where L is the Lagrangian

$$L(\tilde{z}(\tau), \dot{\tilde{z}}(\tau); \tau) = \frac{1}{2\sigma^2} \left[\dot{\tilde{z}}(\tau) - A \right]^2,$$

and the integral is performed (with functional measure $\mathcal{D}[\cdot]$) over the paths $\tilde{z}(\cdot)$ belonging to $\mathcal{C}$, *i.e.* all continuous paths with constraints $\tilde{z}(t') \equiv z'$, $\tilde{z}(t'') \equiv z''$. As carefully discussed in reference [9], a path integral is well defined only if both a continuous formal expression and a discretization rule are given. As done in many applications, the Itô prescription is adopted in the present work.

A first, naive evaluation of the transition probability can be performed *via* Monte Carlo (MC) simulation, by writing equation (6) as

$$p(z''|z') = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_i^n \mathrm{d}g_i \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} \times \exp\left\{ -\frac{1}{2\sigma^2 \Delta t} [z'' - (z_n + A\Delta t)]^2 \right\}, \tag{7}$$

in terms of the variables g_i defined by the relation

$$\mathrm{d}g_k \doteq \frac{\mathrm{d}z_k}{\sqrt{2\pi\sigma^2 \Delta t}} \exp\left\{ -\frac{1}{2\sigma^2 \Delta t} [z_k - (z_{k-1} + A\Delta t)]^2 \right\}, \tag{8}$$

[1] In the expressions for the conditional transition probabilities we omit the times to simplify the notation.

and extracting each g_i from a Gaussian distribution of mean $z_{k-1}+A\Delta t$ and variance $\sigma^2\Delta t$. However, as we will see, this method requires a large number of calls to obtain a good precision. This is due to the fact that each g_i is related to the previous g_{i-1}, so that this implementation of the path integral approach can be seen to be equivalent to a naive MC simulation of random walks, with no variance reduction.

By means of appropriate manipulations [14] of the integrand entering equation (6), it is possible, as shown in the following, to obtain a path integral formula for the transition probability which will contain a factorized integral with a constant kernel and therefore allows a MC implementation with consequent variance reduction. We will refer to this second implementation of the method as path integral with importance sampling.

If we define $y_k \doteq z_k - kA\Delta t$, $k=1,\dots,n$, we can express the transition probability as

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathrm{d}y_1\cdots\mathrm{d}y_n\frac{1}{\sqrt{(2\pi\sigma^2\Delta t)^{n+1}}}\times\exp\left\{-\frac{1}{2\sigma^2\Delta t}\sum_{k=1}^{n+1}[y_k-y_{k-1}]^2\right\},\qquad(9)$$

in order to get rid of the contribution of the drift parameter. Now let us extract from the argument of the exponential function a quadratic form

$$\sum_{k=1}^{n+1}[y_k-y_{k-1}]^2 = y_0^2-2y_1y_0+y_1^2+y_1^2-2y_1y_2+\ldots+y_{n+1}^2 = y^tMy+[y_0^2-2y_1y_0+y_{n+1}^2-2y_ny_{n+1}],\qquad(10)$$

by introducing the n-dimensional array y and the $n\times n$ matrix M defined as

$$y=\begin{pmatrix}y_1\\y_2\\\vdots\\\vdots\\y_n\end{pmatrix},\qquad M=\begin{pmatrix}2 & -1 & 0 & \cdots & \cdots & 0\\-1 & 2 & -1 & 0 & \cdots & 0\\0 & -1 & 2 & -1 & \cdots & 0\\0 & \cdots & -1 & 2 & -1 & 0\\0 & \cdots & \cdots & -1 & 2 & -1\\0 & \cdots & \cdots & \cdots & -1 & 2\end{pmatrix},\qquad(11)$$

where M is a real, symmetric, non singular and tridiagonal matrix. In terms of the eigenvalues m_i of the matrix M, the contribution in equation (10) can be written as

$$y^tMy = w^tO^tMOw = w^tM_dw = \sum_{i=1}^{n}m_iw_i^2,\qquad(12)$$

by introducing the orthogonal matrix O which diagonalizes M, with $w_i = O_{ij}y_j$. Because of the orthogonality of O, the Jacobian of the transformation $y_k\to w_k$ equals 1, so that $\prod_{i=1}^n \mathrm{d}w_i=\prod_{i=1}^n\mathrm{d}y_i$. Thanks to equations (11-12), and after some algebra, it is possible to arrive, as shown in detail in reference [15], at the following expression for the finite-time probability distribution

$$p(z''|z')=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{n}\mathrm{d}h_i\frac{1}{\sqrt{2\pi\sigma^2\Delta t\det(M)}}\times\exp\left\{-\frac{1}{2\sigma^2\Delta t}\left[y_0^2+y_{n+1}^2+\sum_{i=1}^{n}\frac{(y_0O_{1i}+y_{n+1}O_{ni})^2}{m_i}\right]\right\},\qquad(13)$$

where we have introduced new variables h_i which obey the relation

$$\mathrm{d}h_i \doteq \sqrt{\frac{m_i}{2\pi\sigma^2\Delta t}}\times\exp\left\{-\frac{m_i}{2\sigma^2\Delta t}\left[w_i-\frac{(y_0O_{1i}+y_{n+1}O_{ni})}{m_i}\right]^2\right\}\mathrm{d}w_i.\qquad(14)$$

Actually, the probability distribution function, as given by equation (13), is an integral whose kernel is a constant function (with respect to the integration variables) and which can be factorized into the n integrals

$$\int_{-\infty}^{+\infty}\mathrm{d}h_i\exp\left\{-\frac{1}{2\sigma^2\Delta t}\frac{(y_0O_{1i}+y_{n+1}O_{ni})^2}{m_i}\right\},\qquad(15)$$

given in terms of the h_i, which are Gaussian variables that can be extracted from a normal distribution with mean $(y_00_{1i}+y_{n+1}O_{ni})^2/m_i$ and variance $\sigma^2\Delta t/m_i$. Differently to the first, naive implementation of the path integral, now each h_i is no longer dependent on the previous h_{i-1}, and importance sampling over the paths is automatically accounted for. The results of the two realizations of the path integral method here discussed will be compared in Section 3.

2.2 Integration points

Thanks to the method illustrated in Section 2.1, a powerful tool to compute the transition probability in a path integral framework is available and it can be employed if we need to value a generic option with maturity T and with possibility of anticipated exercise at times $t_i=i\Delta t$ ($n\Delta t\doteq T$). As a consequence of this time slicing, one must numerically evaluate $n-1$ mean values of the type (3), in order to check at any time t_i, and for any value of the stock price, whether early exercise is more convenient with respect to holding the option for a future time. To keep under control the computational complexity and the time of execution, it is mandatory to limit as far as possible the number of points for the integral evaluation. This means that we would like to have a linear growth of the number of integration points with the time.

Let us suppose to evaluate each mean value

$$E[\mathcal{O}_i|S_{i-1}] = \int \mathrm{d}z_i p(z_i|z_{i-1})\mathcal{O}_i(\mathrm{e}^{z_i}),$$

with p integration points, *i.e.* considering only p fixed values for z_i. To this end, we can create a grid of possible prices, according to the dynamics of the stochastic process

$$z(t+\Delta t)-z(t)=\ln S(t+\Delta t) - \ln S(t) = A\Delta t + \epsilon\sigma\sqrt{\Delta t}, \quad (16)$$

where ϵ is a random variable following a standardized normal distribution with mean 0 and variance 1.

Starting from z_0, we can thus evaluate the expectation value $E[\mathcal{O}_1|S_0]$ with $p = 2m+1, m \in \mathbb{N}$ values of z_1 centered[2] around the mean value $E[z_1] = z_0 + A\Delta t$ and which differ from each other of a quantity of the order of $\sigma\sqrt{\Delta t}$

$$z_1^j \doteq z_0 + A\Delta t + j\sigma\sqrt{\Delta t}, \qquad j = -m, \ldots, +m.$$

Going on like this, we can evaluate each expectation value $E[\mathcal{O}_2|z_1^j]$ obtained from each one of the z_1's created above with p values for z_2 centered around the mean value

$$E[z_2|z_1^j] = z_1^j + A\Delta t = z_0 + 2A\Delta t + j\sigma\sqrt{\Delta t}.$$

Iterating the procedure until the maturity, we create a deterministic grid of points such that, at a given time t_i, there are $(p-1)i+1$ values of z_i, in agreement with the request of linear growth.

This procedure of selection of the integration points, together with the calculation of the transition probability previously described, is the basis of our path integral simulation of the price of a generic option.

2.3 The limit of continuum and American options

In the case of an American option, the possibility of exercise at any time up to the expiration date allows to develop, within the path integral formalism, a specific algorithm, which, as shown in the following, is precise and very quick.

Given the time slicing considered in Section 2.2, the case of American options requires the limit $\Delta t \to 0$ which, putting $\sigma \to 0$, leads to a delta-like transition probability

$$p(z, t+\Delta t|z_t, t) \approx \delta(z - z_t - A\Delta t).$$

This means that, apart from volatility effects, the price z_i at time t_i will have a value remarkably close to the expected value $\bar{z} \doteq z_{i-1} + A\Delta t$, given by the drift growth. Needless to say, if we should substitute the expression $p(z_i, i\Delta t|z_{i-1}, (i-1)\Delta t) \approx \delta(z_i - \bar{z})$ inside the integrals (3), we would neglect the role of the volatility and consider only a drift growth of the asset prices. In order to take care of the volatility effects, a possible solution is to estimate the integral of interest, *i.e.*

$$E[\mathcal{O}_i|S_{i-1}] = \int_{-\infty}^{+\infty} \mathrm{d}z\, p(z|z_{i-1})\mathcal{O}_i(\mathrm{e}^z), \quad (17)$$

by inserting in equation (17) the analytical expression for the $p(z|z_{i-1})$ transition probability

$$p(z|z_{i-1}) = \frac{1}{\sqrt{2\pi\Delta t\sigma^2}} \exp\left\{-\frac{(z - z_{i-1} - A\Delta t)^2}{2\sigma^2\Delta t}\right\} = \frac{1}{\sqrt{2\pi\Delta t\sigma^2}} \exp\left\{-\frac{(z-\bar{z})^2}{2\sigma^2\Delta t}\right\},$$

together with a Taylor expansion of the kernel function $\mathcal{O}_i(\mathrm{e}^z) \doteq f(z)$ around the expected value $\bar{z}$. Hence, up to the second order in $z - \bar{z}$, the kernel function becomes

$$f(z) = f(\bar{z}) + (z-\bar{z})f'(\bar{z}) + \frac{1}{2}f''(\bar{z})(z-\bar{z})^2 + O((z-\bar{z})^3), \quad (18)$$

which, together with the expression for $p(z|z_{i-1})$, yields

$$E[\mathcal{O}_i|S_{i-1}] = f(\bar{z}) + \frac{\sigma^2}{2}f''(\bar{z}) + \ldots, \quad (19)$$

since the first derivative does not give contribution to equation (17), being the integral of an odd function over the whole z range. The second derivative can be numerically estimated as

$$f''(\bar{z}) = \frac{1}{\delta_\sigma^2}[f(\bar{z}+\delta_\sigma) - 2f(\bar{z}) + f(\bar{z}-\delta_\sigma)], \quad (20)$$

with $\delta_\sigma = O(\sigma\sqrt{\Delta t})$, as dictated by the dynamics of the stochastic process. It is worth noticing that each expectation value $E[\mathcal{O}_i|S_{i-1}]$ can be now computed once $f(\bar{z}) = \mathcal{O}_i(\mathrm{e}^{z_{i-1}+A\Delta t})$ and $f(\bar{z} \pm \delta_\sigma) = \mathcal{O}_i(\mathrm{e}^{z_{i-1}+A\Delta t \pm \delta_\sigma})$ are known. Consequently, if we employ the deterministic grid illustrated in Section 2.2, it is enough to put $p = 3$ to obtain reliable results, provided Δt is taken sufficiently small.

3 Numerical results and discussion

By applying the results derived in Section 2, we have at disposal an efficient path integral algorithm both for the calculation of transition probabilities and the evaluation of option prices. In the present section, the application of the method to European and American options in the BSM model is illustrated and comparisons with the results obtained with the standard procedures known in the literature are shown.

First, the path integral simulation of the probability distribution of the logarithm of the stock prices, $p(\ln S)$, as a function of the logarithm of the stock price, for a BSM-like stochastic model, as given by equation (2), is

[2] Let us recall that between two possible exercise times the probability distribution is Gaussian and it is therefore symmetrical with respect to its mean value.

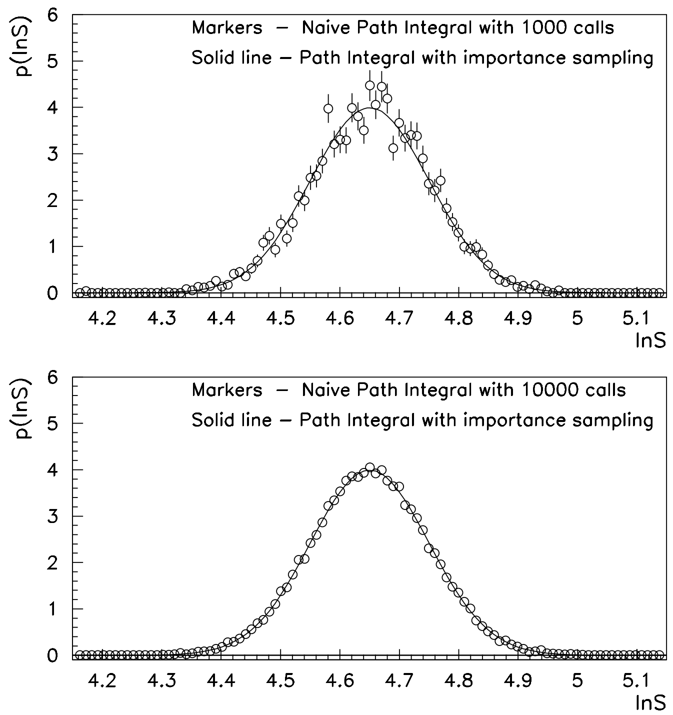

Fig. 1. Simulation of the transition probability distribution in the BSM model as a function of the logarithm of stock prices *via* the two path integral methods discussed in the text: the naive path integral implementation for 10^3 and 10^4 Monte Carlo calls (markers) is compared with the path integral implementation with importance sampling (solid line).

shown in Figure 1. The parameters used in the simulation are: $S_0 = 100$, $X = 110$, $\mu = 0.05$, $\sigma = 0.1$, $t = 0$ year and $T = 1$ year, with 100 time slices. As can be seen, the expected lognormal distribution of the stock prices is correctly reproduced by the path integral numerical simulation.

The plot shows a comparison of the calculation of $p(\ln S)$ as obtained by means of the two path integral algorithms described in Section 2.1. The markers correspond to the naive path integral computation of the probability distribution, without variance reduction, for 10^3 (upper plot) and 10^4 (lower plot) MC iterations. The error bars indicate the 1σ statistical error of the MC calculation. The solid line is the prediction for $p(\ln S)$ as obtained with the path integral simulation with importance sampling. In such a case, only two calls are needed to correctly fit the Gaussian distribution, the numerical error being totally negligible and the algorithm very fast, with a typical time execution of a few seconds on a PentiumIII 500 Mhz PC. On the contrary, the first path integral implementation is much less accurate and CPU time consuming. This is a consequence of the fact that, in the path integral simulation with importance sampling, the presence of a constant integration kernel squeezes to zero the standard estimation error. The diagonalization of the tridiagonal matrix M, which is a basic ingredient of the efficient path integral algorithm developed, is performed according to the standard numerical procedure described in reference [16], realized by means of the routine F02FAF of the NAG program library [17], while the generation of the Gaussian variables h_i follows from the routine RNORML of the CERN program library. It is worth noticing that, by means of the extraction of the random variables h_i, we are creating price paths, since at each intermediate time t_i the asset price is given by

$$S_i = \exp\left\{\sum_{k=1}^{n} O_{ik} h_k + \mathrm{i} A \Delta t\right\}.$$

Therefore, the path integral algorithm can also be applied to the cases in which the derivative to be valued, in the time interval $[0, T]$, has additional constraints, as in the case of interesting path-dependent options, such as barrier options.

Once the transition probability has been computed, the price of an option can be computed in a path integral approach as the conditional expectation value of a given functional of the stochastic process. For example, the price of an European put option, which is considered in the following, will be given by

$$\mathcal{P} = \mathrm{e}^{-r(T-t)} \int_{-\infty}^{+\infty} \mathrm{d}z_f \, p(z_f, T | z_i, t) \max[X - \mathrm{e}^{z_f}, 0],$$

where r is the rik-free interest rate, so that just one-dimensional integrals need to be evaluated. They can be precisely evaluated with standard quadrature rules. In our calculation, the one-dimensional integrals are performed with a standard trapezoidal rule, cross-checked with the routine of adaptive integration D01EAF from the NAG library [17]. A sample of the results obtained for an European put option in the BSM model is shown in Table 1. The predictions of our approach, indicated as path integral, are compared with results available in the literature, as quoted in reference [10]. In Table 1, the entries correspond to the analytical results, the results by binomial trees, and the results of the Green Function Deterministic

Table 1. Price of an European put option in the BSM model for the parameters $t = 0$ year, $T = 0.5$ year, $r = 0.1$, $\sigma = 0.4$, $X = 10$, as a function of different stock prices S_0. 100 time slices are used in the path integral simulation.

S_0	analytical	binomial	GFDNM	path integral
6.0	3.558	3.557	3.557	3.558
8.0	1.918	1.917	1.917	1.918
10.0	0.870	0.866	0.871	0.870
12.0	0.348	0.351	0.349	0.348
14.0	0.128	0.128	0.129	0.128

Table 2. Price of an American put option in the BSM model for the parameters $t = 0$ year, $T = 0.5$ year, $r = 0.1$, $\sigma = 0.4$, $X = 10$, as a function different stock prices S_0. The entries correspond to the following methods: 1. = finite difference; 2. = binomial tree; 3. = GFDNM; 4. = path integral 1; 5. = path integral 2, where the meaning of path integral 1 and path integral 2 is explained in the text. The path integral 1 is performed with 200 time slices and $p = 13$ integration points; the path integral 2 is performed with $\delta_\sigma = 2\sigma\sqrt{\Delta t}$ for numerical differentiation, 300 time slices and $p = 3$.

S_0	1.	2.	3.	4.	5.
6.0	4.00	4.00	4.00	4.00	4.00
8.0	2.095	2.096	2.093	2.095	2.095
10.0	0.921	0.920	0.922	0.922	0.922
12.0	0.362	0.365	0.364	0.362	0.362
14.0	0.132	0.133	0.133	0.132	0.132

Numerical Method (GFDNM) developed in reference [10]. As can be noticed, our results are in perfect agreement with the analytical predictions, while the differences with the other numerical procedures are within the 1% level. The errors in our numbers as due to numerical integration are not specified being well below the digits quoted.

To test the reliability of the sampling over the integration points discussed in Section 2.2 and of the semi-analytical approximation for American options derived in Section 2.3, we present results for the price of an American put option in the BSM model in Table 2, where comparisons with independent results available in the literature are also shown. In Table 2, the results denoted as path integral 1 correspond to procedure of explicit integration over the grid discussed in Section 2.2, while path integral 2 stands for the results obtained with the approximation of the continuum of Section 2.3. As can be seen from Table 2, there is a good agreement of our path integral results with those known in the literature [10] and obtained by means of the binomial tree, of the finite difference method and of the Green function deterministic numerical method (GFDNM). It is worth noticing that our results for the path integral algorithm 1 require only a few seconds on a PentiumIII 500MhZ PC, while the CPU time is negligible for the implementation denoted as path integral 2.

4 Conclusions and prospects

In this paper we have shown how the path integral approach to stochastic processes can be successfully applied to the problem of option pricing in financial analysis. In particular, an efficient implementation of the path integral method has been presented, in order to obtain fast and accurate predictions for a large class of financial derivatives, including those with path-dependent and early exercise features. The key points of the algorithm are a careful evaluation of the transition probability associated to the stochastic model for the time evolution of the asset prices and a suitable choice of the integration points needed to evaluate the quantities of financial interest. Furthermore, a simple and very fast procedure to value American options has been derived, by exploiting the possibility of continuous exercise up to the expiration date.

The results of the path integral algorithm have been carefully compared with those available in the literature for European and American options in the BSM model and found to be in good agreement with the standard numerical procedures used in finance. The method is general and particularly efficient, and it can be easily extended to cope with other financial derivatives (with path-dependent features) and other models beyond the BSM dynamics.

The natural developments of the path integral approach here presented concern the application of the method to value other kinds of quantities of financial interest, for which the analytical solution is not available or not accessible, and the extension of the method of option pricing to more realistic model of the financial dynamics, such as models with stochastic volatility or beyond the BSM Gaussian limit [18–22], in order to search for a better agreement with the real prices as observed in the real market. A further interesting perspective would be to use the path integral algorithm as a benchmark to train neural networks.

These developments are by now under consideration.

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